Coarse geometry of topological groups

Christian Rosendal,
University of Illinois at Chicago

When Geometric Group Theory meets Model Theory,
Marseille, June 2015

A geometric approach to topological groups

The goal of our talk is to present a new geometric approach to the study of Polish and more general topological groups.

A geometric approach to topological groups

The goal of our talk is to present a new geometric approach to the study of Polish and more general topological groups.

As it turns out, this theory will extend geometric group theory of finitely and compactly generated groups and thus permit a full scale import of the vocabulary, tools and problems of that theory to our more general setting.

A geometric approach to topological groups

The goal of our talk is to present a new geometric approach to the study of Polish and more general topological groups.

As it turns out, this theory will extend geometric group theory of finitely and compactly generated groups and thus permit a full scale import of the vocabulary, tools and problems of that theory to our more general setting.

Similarly, our theory generalises geometric non-linear functional analysis and hence provides a common framework for these two hitherto disjoint theories.

A geometric approach to topological groups

The goal of our talk is to present a new geometric approach to the study of Polish and more general topological groups.

As it turns out, this theory will extend geometric group theory of finitely and compactly generated groups and thus permit a full scale import of the vocabulary, tools and problems of that theory to our more general setting.

Similarly, our theory generalises geometric non-linear functional analysis and hence provides a common framework for these two hitherto disjoint theories.

Again, this allows for a unified approach to several similar problems in the two areas.

Uniform spaces

To understand the framework, let us recall A. Weil's concept of uniform spaces.

Uniform spaces

To understand the framework, let us recall A. Weil's concept of uniform spaces.

A uniform space is a set X equipped with a family $\mathcal{U}$ of subsets $E \subseteq X \times X$ called entourages verifying the following conditions.

Uniform spaces

To understand the framework, let us recall A. Weil's concept of uniform spaces.

A uniform space is a set X equipped with a family $\mathcal{U}$ of subsets $E \subseteq X \times X$ called entourages verifying the following conditions.

1. Every $E \in \mathcal{U}$ contains the diagonal $\Delta = \{(x, x) \mid x \in X\}$,

Uniform spaces

To understand the framework, let us recall A. Weil's concept of uniform spaces.

A uniform space is a set X equipped with a family $\mathcal{U}$ of subsets $E \subseteq X \times X$ called entourages verifying the following conditions.

1. Every $E \in \mathcal{U}$ contains the diagonal $\Delta = \{(x, x) \mid x \in X\}$,
2. $\mathcal{U}$ is closed under taking supersets, finite intersections and inverses, $E \mapsto E^{-1} = \{(y, x) \mid (x, y) \in E\}$,

Uniform spaces

To understand the framework, let us recall A. Weil's concept of uniform spaces.

A uniform space is a set X equipped with a family $\mathcal{U}$ of subsets $E \subseteq X \times X$ called entourages verifying the following conditions.

1. Every $E \in \mathcal{U}$ contains the diagonal $\Delta = \{(x,x) \mid x \in X\}$,
2. $\mathcal{U}$ is closed under taking supersets, finite intersections and inverses, $E \mapsto E^{-1} = \{(y,x) \mid (x,y) \in E\}$,
3. for any $E \in \mathcal{U}$, there is $F \in \mathcal{U}$ so that

$$F \circ F = \{(x,z) \mid \exists y \ (x,y),(y,z) \in F\} \subseteq E.$$

Uniform spaces

To understand the framework, let us recall A. Weil's concept of uniform spaces.

A uniform space is a set X equipped with a family $\mathcal{U}$ of subsets $E \subseteq X \times X$ called entourages verifying the following conditions.

1. Every $E \in \mathcal{U}$ contains the diagonal $\Delta = \{(x,x) \mid x \in X\}$,
2. $\mathcal{U}$ is closed under taking supersets, finite intersections and inverses, $E \mapsto E^{-1} = \{(y,x) \mid (x,y) \in E\}$,
3. for any $E \in \mathcal{U}$, there is $F \in \mathcal{U}$ so that

$$F \circ F = \{(x,z) \mid \exists y \ (x,y),(y,z) \in F\} \subseteq E.$$

A uniform space is intended to capture the idea of being uniformly close in a topological space and hence gives rise to concepts of Cauchy sequences and completeness.

Pseudometric spaces

The canonical example of a uniform space is when (X, d) is a metric or, more generally, a pseudometric space.

Pseudometric spaces

The canonical example of a uniform space is when (X, d) is a metric or, more generally, a pseudometric space.

Recall here that a pseudometric space is a set X equipped with an écart.

Pseudometric spaces

The canonical example of a uniform space is when (X, d) is a metric or, more generally, a pseudometric space.

Recall here that a pseudometric space is a set X equipped with an écart.

In this case, we may, for every $\alpha > 0$, set

$$E_\alpha = \{(x, y) \mid d(x, y) < \alpha\}$$

Pseudometric spaces

The canonical example of a uniform space is when (X, d) is a metric or, more generally, a pseudometric space.

Recall here that a pseudometric space is a set X equipped with an écart.

In this case, we may, for every $\alpha > 0$, set

$$E_\alpha = \{(x, y) \mid d(x, y) < \alpha\}$$

and define a uniformity $\mathcal{U}_d$ by

$$\mathcal{U}_d = \{E \subseteq X \times X \mid \exists \alpha > 0 \;\; E_\alpha \subseteq E\}.$$

J. Roe's Coarse spaces

A coarse space is a set X equipped with a collection $\mathcal{E}$ of subsets $E \subseteq X \times X$ called entourages satisfying the following conditions.

J. Roe's Coarse spaces

A coarse space is a set X equipped with a collection $\mathcal{E}$ of subsets $E \subseteq X \times X$ called entourages satisfying the following conditions.

1. The diagonal Δ belongs to $\mathcal{E}$,

J. Roe's Coarse spaces

A coarse space is a set X equipped with a collection $\mathcal{E}$ of subsets $E \subseteq X \times X$ called entourages satisfying the following conditions.

1. The diagonal Δ belongs to $\mathcal{E}$,
2. if $E \subseteq F \in \mathcal{E}$, then also $E \in \mathcal{E}$,

J. Roe's Coarse spaces

A coarse space is a set X equipped with a collection $\mathcal{E}$ of subsets $E \subseteq X \times X$ called entourages satisfying the following conditions.

1. The diagonal Δ belongs to $\mathcal{E}$,
2. if $E \subseteq F \in \mathcal{E}$, then also $E \in \mathcal{E}$,
3. if $E, F \in \mathcal{E}$, then $E \cup F, E^{-1}, E \circ F \in \mathcal{E}$.

J. Roe's Coarse spaces

A coarse space is a set X equipped with a collection $\mathcal{E}$ of subsets $E \subseteq X \times X$ called entourages satisfying the following conditions.

1. The diagonal Δ belongs to $\mathcal{E}$,
2. if $E \subseteq F \in \mathcal{E}$, then also $E \in \mathcal{E}$,
3. if $E, F \in \mathcal{E}$, then $E \cup F, E^{-1}, E \circ F \in \mathcal{E}$.

Again, if (X, d) is a pseudometric space, there is a canonical coarse structure $\mathcal{E}_d$ obtained by

$$\mathcal{E}_d = \{E \subseteq X \times X \mid \exists \alpha < \infty \ \ E \subseteq E_\alpha\}.$$

J. Roe's Coarse spaces

A coarse space is a set X equipped with a collection $\mathcal{E}$ of subsets $E \subseteq X \times X$ called entourages satisfying the following conditions.

1. The diagonal Δ belongs to $\mathcal{E}$,
2. if $E \subseteq F \in \mathcal{E}$, then also $E \in \mathcal{E}$,
3. if $E, F \in \mathcal{E}$, then $E \cup F, E^{-1}, E \circ F \in \mathcal{E}$.

Again, if (X, d) is a pseudometric space, there is a canonical coarse structure $\mathcal{E}_d$ obtained by

$$\mathcal{E}_d = \{E \subseteq X \times X \bigm| \exists \alpha < \infty \;\; E \subseteq E_\alpha\}.$$

The main point here is that, for a uniform structure, we are interested in E_α for α small, but positive,

J. Roe's Coarse spaces

A coarse space is a set X equipped with a collection $\mathcal{E}$ of subsets $E \subseteq X \times X$ called entourages satisfying the following conditions.

1. The diagonal Δ belongs to $\mathcal{E}$,
2. if $E \subseteq F \in \mathcal{E}$, then also $E \in \mathcal{E}$,
3. if $E, F \in \mathcal{E}$, then $E \cup F, E^{-1}, E \circ F \in \mathcal{E}$.

Again, if (X, d) is a pseudometric space, there is a canonical coarse structure $\mathcal{E}_d$ obtained by

$$\mathcal{E}_d = \{E \subseteq X \times X \mid \exists \alpha < \infty \;\; E \subseteq E_\alpha\}.$$

The main point here is that, for a uniform structure, we are interested in E_α for α small, but positive, while, for a coarse structure, α is often large, but finite.

Left-uniform structure on a topological group

If G is a topological group, its left-uniformity $\mathcal{U}_L$ is that generated by entourages of the form

$$E_V = \{(x, y) \in G \times G \mid x^{-1}y \in V\},$$

where V varies over all identity neighbourhoods in G.

Left-uniform structure on a topological group

If G is a topological group, its left-uniformity $\mathcal{U}_L$ is that generated by entourages of the form

$$E_V = \{(x, y) \in G \times G \mid x^{-1}y \in V\},$$

where V varies over all identity neighbourhoods in G.

A basic theorem, due essentially to G. Birkhoff (fils) and S. Kakutani, is that

$$\mathcal{U}_L = \bigcup_d \mathcal{U}_d,$$

where the union is taken over all continuous left-invariant écarts d on G, i.e., so that $d(zx, zy) = d(x, y)$.

Left-coarse structure on a topological group

Now, coarse structures should be viewed as dual to uniform structures, so we obtain appropriate definitions by placing negations strategically in definitions for concepts of uniformities.

Left-coarse structure on a topological group

Now, coarse structures should be viewed as dual to uniform structures, so we obtain appropriate definitions by placing negations strategically in definitions for concepts of uniformities.

Definition

If G is a topological group, its left-coarse structure $\mathcal{E}_L$ is given by

$$\mathcal{E}_L = \bigcap_{d} \mathcal{E}_d,$$

where the intersection is taken over all continuous left-invariant écarts d on G.

Relatively OB sets

The definition of the coarse structure $\mathcal{E}_L$ is not immediately transparent and it is thus useful to have alternate descriptions of it.

Relatively OB sets

The definition of the coarse structure $\mathcal{E}_L$ is not immediately transparent and it is thus useful to have alternate descriptions of it.

Definition

A subset $A \subseteq G$ of a topological group is said to be relatively (OB) in G if

$$\mathrm{diam}_d(A) < \infty$$

for every continuous left-invariant écart d on G.

Relatively OB sets

The definition of the coarse structure $\mathcal{E}_L$ is not immediately transparent and it is thus useful to have alternate descriptions of it.

Definition

A subset $A \subseteq G$ of a topological group is said to be relatively (OB) in G if

$$\mathrm{diam}_d(A) < \infty$$

for every continuous left-invariant écart d on G.

One may easily show that the class $\mathcal{OB}$ of relatively (OB) subsets is an ideal of subsets of G stable under the operations

$$A \mapsto A^{-1}, \quad (A, B) \mapsto AB \quad \text{and} \quad A \mapsto \overline{A}.$$

Proposition

The left-coarse structure $\mathcal{E}_L$ on a topological group G is generated by entourages of the form

$$E_A = \{(x, y) \mid x^{-1}y \in A\},$$

where $A \in \mathcal{OB}$.

Proposition

The left-coarse structure $\mathcal{E}_L$ on a topological group G is generated by entourages of the form

$$E_A = \{(x, y) \mid x^{-1}y \in A\},$$

where $A \in \mathcal{OB}$.

For simplicity, we focus on an extension of the class of Polish groups, encompassing all Banach spaces and locally compact σ-compact groups.

Proposition

The left-coarse structure $\mathcal{E}_L$ on a topological group G is generated by entourages of the form

$$E_A = \{(x, y) \mid x^{-1}y \in A\},$$

where $A \in \mathcal{OB}$.

For simplicity, we focus on an extension of the class of Polish groups, encompassing all Banach spaces and locally compact σ-compact groups.

Definition

A topological group G is European if it is Baire and is countably generated over every identity neighbourhood,

Proposition

The left-coarse structure $\mathcal{E}_L$ on a topological group G is generated by entourages of the form

$$E_A = \{(x, y) \mid x^{-1}y \in A\},$$

where $A \in \mathcal{OB}$.

For simplicity, we focus on an extension of the class of Polish groups, encompassing all Banach spaces and locally compact σ-compact groups.

Definition

A topological group G is European if it is Baire and is countably generated over every identity neighbourhood, i.e., for every $V \ni 1$ open, there is a countable set $D \subseteq G$ so that $G = \langle D \cup V \rangle$.

By the mechanics of the Birkhoff–Kakutani metrisation theorem, we have the following description of the relatively (OB) sets.

By the mechanics of the Birkhoff–Kakutani metrisation theorem, we have the following description of the relatively (OB) sets.

Proposition

A subset A of a European topological group G is relatively (OB) if and only if, for every identity neighbourhood V, there are a finite set $F \subseteq G$ and $k \geqslant 1$ so that

$$A \subseteq (FV)^k.$$

By the mechanics of the Birkhoff–Kakutani metrisation theorem, we have the following description of the relatively (OB) sets.

Proposition

A subset A of a European topological group G is relatively (OB) if and only if, for every identity neighbourhood V, there are a finite set $F \subseteq G$ and $k \geqslant 1$ so that

$$A \subseteq (FV)^k.$$

• For example, the relatively (OB) subsets of a countable discrete group are simply the finite sets.

By the mechanics of the Birkhoff–Kakutani metrisation theorem, we have the following description of the relatively (OB) sets.

Proposition

A subset A of a European topological group G is relatively (OB) if and only if, for every identity neighbourhood V, there are a finite set $F \subseteq G$ and $k \geqslant 1$ so that

$$A \subseteq (FV)^k.$$

• For example, the relatively (OB) subsets of a countable discrete group are simply the finite sets.

• More generally, in a locally compact σ-compact group, they are the relatively compact subsets.

By the mechanics of the Birkhoff–Kakutani metrisation theorem, we have the following description of the relatively (OB) sets.

Proposition

A subset A of a European topological group G is relatively (OB) if and only if, for every identity neighbourhood V, there are a finite set $F \subseteq G$ and $k \geqslant 1$ so that

$$A \subseteq (FV)^k.$$

• For example, the relatively (OB) subsets of a countable discrete group are simply the finite sets.

• More generally, in a locally compact σ-compact group, they are the relatively compact subsets.

• Similarly, in the underlying additive group $(X, +)$ of a Banach space $(X, \|\cdot\|)$, they are the norm bounded subsets.

Metrisability

As with the topology and left-uniformity on a topological group, metrisability of the left-coarse structure is not automatic.

Metrisability

As with the topology and left-uniformity on a topological group, metrisability of the left-coarse structure is not automatic.

Here a coarse space $(X, \mathcal{E})$ is metrisable if there is a metric d on X so that $\mathcal{E} = \mathcal{E}_d$.

Metrisability

As with the topology and left-uniformity on a topological group, metrisability of the left-coarse structure is not automatic.

Here a coarse space $(X, \mathcal{E})$ is metrisable if there is a metric d on X so that $\mathcal{E} = \mathcal{E}_d$.

Theorem

The following conditions are equivalent for a European group G.

Metrisability

As with the topology and left-uniformity on a topological group, metrisability of the left-coarse structure is not automatic.

Here a coarse space $(X, \mathcal{E})$ is metrisable if there is a metric d on X so that $\mathcal{E} = \mathcal{E}_d$.

Theorem

The following conditions are equivalent for a European group G.

1. *The left-coarse structure $\mathcal{E}_L$ is metrisable,*

Metrisability

As with the topology and left-uniformity on a topological group, metrisability of the left-coarse structure is not automatic.

Here a coarse space $(X, \mathcal{E})$ is metrisable if there is a metric d on X so that $\mathcal{E} = \mathcal{E}_d$.

Theorem

The following conditions are equivalent for a European group G.

1. *The left-coarse structure $\mathcal{E}_L$ is metrisable,*
2. *there is a continuous left-invariant écart d on G so that $\mathcal{E}_L = \mathcal{E}_d$,*

Metrisability

As with the topology and left-uniformity on a topological group, metrisability of the left-coarse structure is not automatic.

Here a coarse space $(X, \mathcal{E})$ is metrisable if there is a metric d on X so that $\mathcal{E} = \mathcal{E}_d$.

Theorem

The following conditions are equivalent for a European group G.

1. *The left-coarse structure $\mathcal{E}_L$ is metrisable,*
2. *there is a continuous left-invariant écart d on G so that $\mathcal{E}_L = \mathcal{E}_d$,*
3. *G is locally (OB),*

Metrisability

As with the topology and left-uniformity on a topological group, metrisability of the left-coarse structure is not automatic.

Here a coarse space $(X, \mathcal{E})$ is metrisable if there is a metric d on X so that $\mathcal{E} = \mathcal{E}_d$.

Theorem

The following conditions are equivalent for a European group G.

1. *The left-coarse structure $\mathcal{E}_L$ is metrisable,*
2. *there is a continuous left-invariant écart d on G so that $\mathcal{E}_L = \mathcal{E}_d$,*
3. *G is locally (OB), i.e., there is a relatively (OB) identity neighbourhood $V \subseteq G$.*

In case d is a continuous left-invariant écart inducing the coarse structure on G, that is, $\mathcal{E}_L = \mathcal{E}_d$, we say that d is coarsely proper.

In case d is a continuous left-invariant écart inducing the coarse structure on G, that is, $\mathcal{E}_L = \mathcal{E}_d$, we say that d is coarsely proper.

Thus, d is coarsely proper if and only if the finite d-diameter subsets of G are simply the relatively (OB) sets.

In case d is a continuous left-invariant écart inducing the coarse structure on G, that is, $\mathcal{E}_L = \mathcal{E}_d$, we say that d is coarsely proper.

Thus, d is coarsely proper if and only if the finite d-diameter subsets of G are simply the relatively (OB) sets.

Alternatively, we may quasiorder the continuous left-invariant écarts on G by

$$\partial \lll d \quad \Leftrightarrow$$

In case d is a continuous left-invariant écart inducing the coarse structure on G, that is, $\mathcal{E}_L = \mathcal{E}_d$, we say that d is coarsely proper.

Thus, d is coarsely proper if and only if the finite d-diameter subsets of G are simply the relatively (OB) sets.

Alternatively, we may quasiorder the continuous left-invariant écarts on G by

$$\partial \lll d \quad \Leftrightarrow \quad \exists \rho\colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ so that } \partial(x,y) \leqslant \rho\big(d(x,y)\big).$$

In case d is a continuous left-invariant écart inducing the coarse structure on G, that is, $\mathcal{E}_L = \mathcal{E}_d$, we say that d is coarsely proper.

Thus, d is coarsely proper if and only if the finite d-diameter subsets of G are simply the relatively (OB) sets.

Alternatively, we may quasiorder the continuous left-invariant écarts on G by

$$\partial \lll d \quad \Leftrightarrow \quad \exists \rho\colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ so that } \partial(x, y) \leqslant \rho\big(d(x, y)\big).$$

The coarsely proper écarts are then the maximal elements in this ordering.

In case d is a continuous left-invariant écart inducing the coarse structure on G, that is, $\mathcal{E}_L = \mathcal{E}_d$, we say that d is coarsely proper.

Thus, d is coarsely proper if and only if the finite d-diameter subsets of G are simply the relatively (OB) sets.

Alternatively, we may quasiorder the continuous left-invariant écarts on G by

$$\partial \lll d \quad \Leftrightarrow \quad \exists \rho\colon \mathbb{R}_+ \to \mathbb{R}_+ \text{ so that } \partial(x,y) \leqslant \rho\big(d(x,y)\big).$$

The coarsely proper écarts are then the maximal elements in this ordering.

The previous theorem can be seen as an extension of a result due to S. Kakutani and K. Kodaira stating that any locally compact σ-compact group carries a continuous left-invariant proper écart, i.e., so that balls are compact.

Quasimetric spaces

The previous definitions and results identifies a canonical coarse geometry of every topological group G, that for locally (OB) European groups is given by a coarsely proper écart.

Quasimetric spaces

The previous definitions and results identifies a canonical coarse geometry of every topological group G, that for locally (OB) European groups is given by a coarsely proper écart.

However, in several classical cases, an even stronger canonical geometry can be detected, i.e., a geometry distinguishing finer features of the spaces/groups.

Quasimetric spaces

The previous definitions and results identifies a canonical coarse geometry of every topological group G, that for locally (OB) European groups is given by a coarsely proper écart.

However, in several classical cases, an even stronger canonical geometry can be detected, i.e., a geometry distinguishing finer features of the spaces/groups.

Definition

A map $\phi\colon (M, d_M) \to (N, d_N)$ between pseudometric spaces is said to be a quasi-isometric embedding if there are constants K and C so that

$$\frac{1}{K} \cdot d_M(x, y) - C \leqslant d_N(\phi x, \phi y) \leqslant K \cdot d_M(x, y) + C.$$

Quasimetric spaces

The previous definitions and results identifies a canonical coarse geometry of every topological group G, that for locally (OB) European groups is given by a coarsely proper écart.

However, in several classical cases, an even stronger canonical geometry can be detected, i.e., a geometry distinguishing finer features of the spaces/groups.

Definition

A map $\phi\colon (M, d_M) \to (N, d_N)$ between pseudometric spaces is said to be a quasi-isometric embedding if there are constants K and C so that

$$\frac{1}{K} \cdot d_M(x, y) - C \ \leqslant\ d_N(\phi x, \phi y) \ \leqslant\ K \cdot d_M(x, y) + C.$$

Moreover, ϕ is a quasi-isometry if in addition $\phi[M]$ is cobounded in N, that is, $\sup_{y \in N} d_N(y, \phi[M]) < \infty$.

Example: Finitely generated groups

Consider a finitely generated group Γ and fix a finite symmetric generating set $S \subseteq \Gamma$.

Example: Finitely generated groups

Consider a finitely generated group Γ and fix a finite symmetric generating set $S \subseteq \Gamma$.

We may define a length function on Γ by letting

$$\ell_S(x) = \min\big(k \;\big|\; \exists s_1, \ldots, s_k \in S \colon x = s_1 \cdots s_k\big).$$

Example: Finitely generated groups

Consider a finitely generated group Γ and fix a finite symmetric generating set $S \subseteq \Gamma$.

We may define a length function on Γ by letting

$$\ell_S(x) = \min\big(k \;\big|\; \exists s_1, \ldots, s_k \in S \colon x = s_1 \cdots s_k\big).$$

From this we define a left-invariant metric, called the word metric, on Γ by

$$\rho_S(x, y) = \ell_S(x^{-1}y).$$

Example: Finitely generated groups

Consider a finitely generated group Γ and fix a finite symmetric generating set $S \subseteq \Gamma$.

We may define a length function on Γ by letting

$$\ell_S(x) = \min\big(k \;\big|\; \exists s_1, \ldots, s_k \in S \colon x = s_1 \cdots s_k\big).$$

From this we define a left-invariant metric, called the word metric, on Γ by

$$\rho_S(x, y) = \ell_S(x^{-1}y).$$

The fundamental observation underlying geometric group theory is then that given any two finite symmetric generating sets $S, S' \subseteq \Gamma$, the word metrics ρ_S and $\rho_{S'}$ are quasi-isometric,

Example: Finitely generated groups

Consider a finitely generated group Γ and fix a finite symmetric generating set $S \subseteq \Gamma$.

We may define a length function on Γ by letting

$$\ell_S(x) = \min(k \mid \exists s_1, \ldots, s_k \in S \colon x = s_1 \cdots s_k).$$

From this we define a left-invariant metric, called the word metric, on Γ by

$$\rho_S(x, y) = \ell_S(x^{-1}y).$$

The fundamental observation underlying geometric group theory is then that given any two finite symmetric generating sets $S, S' \subseteq \Gamma$, the word metrics ρ_S and $\rho_{S'}$ are quasi-isometric, i.e.,

$$\mathrm{id} \colon (\Gamma, \rho_S) \to (\Gamma, \rho_{S'}) \quad \text{is a quasi-isometry.}$$

Quasimetric structure

To generalise the example of finitely generated groups, we refine the quasiordering $\lll$ of continuous left-invariant écarts as follows.

Quasimetric structure

To generalise the example of finitely generated groups, we refine the quasiordering $\lll$ of continuous left-invariant écarts as follows.

$$\partial \ll d \quad \Leftrightarrow \quad \exists K, C \quad \partial \leqslant K \cdot d + C.$$

To generalise the example of finitely generated groups, we refine the quasiordering $\lll$ of continuous left-invariant écarts as follows.

$$\partial \ll d \quad \Leftrightarrow \quad \exists K, C \quad \partial \leqslant K \cdot d + C.$$

The maximal elements in this ordering (provided they exist) are called maximal écarts on G.

Quasimetric structure

To generalise the example of finitely generated groups, we refine the quasiordering $\lll$ of continuous left-invariant écarts as follows.

$$\partial \ll d \quad \Leftrightarrow \quad \exists K, C \quad \partial \leqslant K \cdot d + C.$$

The maximal elements in this ordering (provided they exist) are called maximal écarts on G.

Since $\ll$ refines $\lll$, every maximal écart is automatically coarsely proper.

Quasimetric structure

To generalise the example of finitely generated groups, we refine the quasiordering $\lll$ of continuous left-invariant écarts as follows.

$$\partial \ll d \quad \Leftrightarrow \quad \exists K, C \quad \partial \leqslant K \cdot d + C.$$

The maximal elements in this ordering (provided they exist) are called maximal écarts on G.

Since $\ll$ refines $\lll$, every maximal écart is automatically coarsely proper.

Also, any two maximal écarts are necessarily quasi-isometric and thus provide a canonical and well-defined quasimetric structure on G,

Quasimetric structure

To generalise the example of finitely generated groups, we refine the quasiordering $\lll$ of continuous left-invariant écarts as follows.

$$\partial \ll d \quad \Leftrightarrow \quad \exists K, C \quad \partial \leqslant K \cdot d + C.$$

The maximal elements in this ordering (provided they exist) are called maximal écarts on G.

Since $\ll$ refines $\lll$, every maximal écart is automatically coarsely proper.

Also, any two maximal écarts are necessarily quasi-isometric and thus provide a canonical and well-defined quasimetric structure on G, that is, a space with a quasi-isometric equivalence class of écarts.

Proposition

The following are equivalent for a continuous left-invariant écart d on a topological group G.

Proposition

The following are equivalent for a continuous left-invariant écart d on a topological group G.

1. *d is maximal,*

Proposition

The following are equivalent for a continuous left-invariant écart d on a topological group G.

1. *d is maximal,*
2. *there is a relatively (OB) subset $A \subseteq G$ algebraically generating G so that d and ρ_A are quasi-isometric.*

Proposition

The following are equivalent for a continuous left-invariant écart d on a topological group G.

1. *d is maximal,*
2. *there is a relatively (OB) subset $A \subseteq G$ algebraically generating G so that d and ρ_A are quasi-isometric.*

Theorem

A European group G admits a maximal écart if and only if G is (OB) generated,

Proposition

The following are equivalent for a continuous left-invariant écart d on a topological group G.

1. d *is maximal,*
2. *there is a relatively (OB) subset* $A \subseteq G$ *algebraically generating* G *so that* d *and* ρ_A *are quasi-isometric.*

Theorem

A European group G *admits a maximal écart if and only if* G *is (OB) generated, that is, there is a relatively (OB) subset* $A \subseteq G$ *algebraically generating* G.

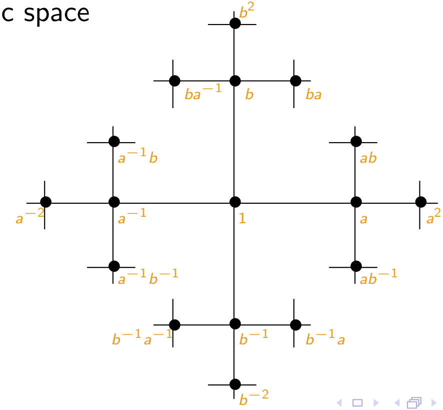
Examples

The word metric ρ_S of a finite symmetric generating set S of a discrete group Γ is maximal and thus identifies the quasimetric structure of Γ as a topological group.

Examples

The word metric ρ_S of a finite symmetric generating set S of a discrete group Γ is maximal and thus identifies the quasimetric structure of Γ as a topological group.

For example, the free non-abelian group $\mathbb{F}_2$ on two generators a, b gives rise to the quasimetric space



Similarly, every compactly generated locally compact group admits a continuous left-invariant écart quasi-isometric to the word metric of a compact generating set.

Similarly, every compactly generated locally compact group admits a continuous left-invariant écart quasi-isometric to the word metric of a compact generating set.

Proposition

Any continuous left-invariant geodesic écart is maximal.

Similarly, every compactly generated locally compact group admits a continuous left-invariant écart quasi-isometric to the word metric of a compact generating set.

Proposition

Any continuous left-invariant geodesic écart is maximal.

Thus, the norm-metric $d_{\|\cdot\|}$ on a Banach space $(X, \|\cdot\|)$ is maximal.

Similarly, every compactly generated locally compact group admits a continuous left-invariant écart quasi-isometric to the word metric of a compact generating set.

Proposition

Any continuous left-invariant geodesic écart is maximal.

Thus, the norm-metric $d_{\|\cdot\|}$ on a Banach space $(X, \|\cdot\|)$ is maximal.

From these examples we see that the theory presented is a conservative extension of geometric group theory for finitely or compactly generated groups and of the geometric non-linear analysis of Banach spaces.

Homeomorphism groups

Let M be a compact manifold and $\mathcal{V} = \{V_i\}_{i=1}^k$ an open covering of M.

Homeomorphism groups

Let M be a compact manifold and $\mathcal{V} = \{V_i\}_{i=1}^k$ an open covering of M.

By fundamental work of Edwards and Kirby, there is an identity neighbourhood U in $\mathrm{Homeo}(M)$ so that every element $h \in U$ can be written as $h = g_1 \cdots g_k$, where

$$\mathrm{supp}(g_i) \subseteq V_i.$$

Homeomorphism groups

Let M be a compact manifold and $\mathcal{V} = \{V_i\}_{i=1}^k$ an open covering of M.

By fundamental work of Edwards and Kirby, there is an identity neighbourhood U in $\mathrm{Homeo}(M)$ so that every element $h \in U$ can be written as $h = g_1 \cdots g_k$, where

$$\mathrm{supp}(g_i) \subseteq V_i.$$

We may thus define the corresponding fragmentation norm on the identity component $\mathrm{Homeo}_0(M)$ of isotopically trivial homeomorphisms by letting

$$\ell_{\mathcal{V}}(h) = \min\big(m \bigm| h = g_1 \cdots g_m \ \& \ \mathrm{supp}(g_i) \subseteq V_{j_i} \text{ for some } j_i\big).$$

Homeomorphism groups

Let M be a compact manifold and $\mathcal{V} = \{V_i\}_{i=1}^k$ an open covering of M.

By fundamental work of Edwards and Kirby, there is an identity neighbourhood U in $\mathrm{Homeo}(M)$ so that every element $h \in U$ can be written as $h = g_1 \cdots g_k$, where

$$\mathrm{supp}(g_i) \subseteq V_i.$$

We may thus define the corresponding fragmentation norm on the identity component $\mathrm{Homeo}_0(M)$ of isotopically trivial homeomorphisms by letting

$$\ell_{\mathcal{V}}(h) = \min\big(m \,\big|\, h = g_1 \cdots g_m \;\&\; \mathrm{supp}(g_i) \subseteq V_{j_i} \text{ for some } j_i\big).$$

From this, we obtain a left-invariant metric by

$$\rho_{\mathcal{V}}(g, f) = \ell_{\mathcal{V}}(g^{-1} f).$$

Proposition (K. Mann & C.R.)

For all sufficiently fine open covers $\mathcal{V}$ of a compact manifold M, the metric $\rho_{\mathcal{V}}$ is quasi-isometric to a maximal metric on $\mathrm{Homeo}_0(M)$.

Proposition (K. Mann & C.R.)

For all sufficiently fine open covers $\mathcal{V}$ of a compact manifold M, the metric $\rho_{\mathcal{V}}$ is quasi-isometric to a maximal metric on $\mathrm{Homeo}_0(M)$.

We should mention that, in the case of compact surfaces M, E. Militon in previous work has been able to explicitly describe this maximal metric as the maximal displacement metric on the universal cover $\tilde{M}$.

Proposition (K. Mann & C.R.)

For all sufficiently fine open covers $\mathcal{V}$ of a compact manifold M, the metric $\rho_{\mathcal{V}}$ is quasi-isometric to a maximal metric on $\mathrm{Homeo}_0(M)$.

We should mention that, in the case of compact surfaces M, E. Militon in previous work has been able to explicitly describe this maximal metric as the maximal displacement metric on the universal cover $\tilde{M}$.

Theorem

For all $n \geqslant 1$, $\mathrm{Homeo}_0(S^n)$ is quasi-isometric to a point.

Proposition (K. Mann & C.R.)

For all sufficiently fine open covers $\mathcal{V}$ of a compact manifold M, the metric $\rho_{\mathcal{V}}$ is quasi-isometric to a maximal metric on $\mathrm{Homeo}_0(M)$.

We should mention that, in the case of compact surfaces M, E. Militon in previous work has been able to explicitly describe this maximal metric as the maximal displacement metric on the universal cover $\tilde{M}$.

Theorem

For all $n \geqslant 1$, $\mathrm{Homeo}_0(S^n)$ is quasi-isometric to a point.

Theorem (K. Mann and C.R.)

Let M be a compact manifold of dimension $\geqslant 2$ so that $\pi_1(M)$ contains an element of infinite order.

Proposition (K. Mann & C.R.)

For all sufficiently fine open covers $\mathcal{V}$ of a compact manifold M, the metric $\rho_{\mathcal{V}}$ is quasi-isometric to a maximal metric on $\mathrm{Homeo}_0(M)$.

We should mention that, in the case of compact surfaces M, E. Militon in previous work has been able to explicitly describe this maximal metric as the maximal displacement metric on the universal cover $\tilde{M}$.

Theorem

For all $n \geqslant 1$, $\mathrm{Homeo}_0(S^n)$ is quasi-isometric to a point.

Theorem (K. Mann and C.R.)

Let M be a compact manifold of dimension $\geqslant 2$ so that $\pi_1(M)$ contains an element of infinite order. Then there is a quasi-isometric isomorphic embedding of the Banach space $C([0,1])$ into $\mathrm{Homeo}_0(M)$.

Proposition (K. Mann & C.R.)

For all sufficiently fine open covers $\mathcal{V}$ of a compact manifold M, the metric $\rho_{\mathcal{V}}$ is quasi-isometric to a maximal metric on $\mathrm{Homeo}_0(M)$.

We should mention that, in the case of compact surfaces M, E. Militon in previous work has been able to explicitly describe this maximal metric as the maximal displacement metric on the universal cover $\tilde{M}$.

Theorem

For all $n \geqslant 1$, $\mathrm{Homeo}_0(S^n)$ is quasi-isometric to a point.

Theorem (K. Mann and C.R.)

Let M be a compact manifold of dimension $\geqslant 2$ so that $\pi_1(M)$ contains an element of infinite order. Then there is a quasi-isometric isomorphic embedding of the Banach space $C([0,1])$ into $\mathrm{Homeo}_0(M)$.
In particular, every separable metric space admits a quasi-isometric embedding into $\mathrm{Homeo}_0(M)$.

Applications to model theory

To ensure a proper translation between properties of a countable first-order structure $\mathbf{A}$ and its automorphism group, we shall work under the relatively mild assumption that $\mathbf{A}$ is ω-homogeneous.

Applications to model theory

To ensure a proper translation between properties of a countable first-order structure $\mathbf{A}$ and its automorphism group, we shall work under the relatively mild assumption that $\mathbf{A}$ is ω-homogeneous.

That is, for all finite tuples $\bar{a}$ and $\bar{b}$ in $\mathbf{A}$,

$$\mathcal{O}(\bar{a}) = \mathcal{O}(\bar{b}) \;\Leftrightarrow\; \mathrm{tp}^{\mathbf{A}}(\bar{a}) = \mathrm{tp}^{\mathbf{A}}(\bar{b}),$$

where $\mathcal{O}(\bar{a})$ denotes the orbit of $\bar{a}$ under the action of $\mathrm{Aut}(\mathbf{A})$ on $\mathbf{A}^{|\bar{a}|}$.

Applications to model theory

To ensure a proper translation between properties of a countable first-order structure $\mathbf{A}$ and its automorphism group, we shall work under the relatively mild assumption that $\mathbf{A}$ is ω-homogeneous.

That is, for all finite tuples $\bar{a}$ and $\bar{b}$ in $\mathbf{A}$,

$$\mathcal{O}(\bar{a}) = \mathcal{O}(\bar{b}) \;\Leftrightarrow\; \mathrm{tp}^{\mathbf{A}}(\bar{a}) = \mathrm{tp}^{\mathbf{A}}(\bar{b}),$$

where $\mathcal{O}(\bar{a})$ denotes the orbit of $\bar{a}$ under the action of $\mathrm{Aut}(\mathbf{A})$ on $\mathbf{A}^{|\bar{a}|}$.

So assume $\mathbf{A}$ is a countable ω-homogeneous structure, $\bar{a}$ is a finite tuple in $\mathbf{A}$ and $\mathcal{S}$ is a finite collection of parameter-free complete types on $\mathbf{A}$.

Applications to model theory

To ensure a proper translation between properties of a countable first-order structure $\mathbf{A}$ and its automorphism group, we shall work under the relatively mild assumption that $\mathbf{A}$ is ω-homogeneous.

That is, for all finite tuples $\overline{a}$ and $\overline{b}$ in $\mathbf{A}$,

$$\mathcal{O}(\overline{a}) = \mathcal{O}(\overline{b}) \;\Leftrightarrow\; \mathrm{tp}^{\mathbf{A}}(\overline{a}) = \mathrm{tp}^{\mathbf{A}}(\overline{b}),$$

where $\mathcal{O}(\overline{a})$ denotes the orbit of $\overline{a}$ under the action of $\mathrm{Aut}(\mathbf{A})$ on $\mathbf{A}^{|\overline{a}|}$.

So assume $\mathbf{A}$ is a countable ω-homogeneous structure, $\overline{a}$ is a finite tuple in $\mathbf{A}$ and $\mathcal{S}$ is a finite collection of parameter-free complete types on $\mathbf{A}$.

$\mathbf{X}_{\overline{a},\mathcal{S}}$ is the graph on $\mathcal{O}(\overline{a})$ obtained by connecting distinct $\overline{b}, \overline{c} \in \mathcal{O}(\overline{a})$ by an edge if and only if

$$\mathrm{tp}^{\mathbf{A}}(\overline{b}, \overline{c}) \in \mathcal{S} \quad \text{or} \quad \mathrm{tp}^{\mathbf{A}}(\overline{c}, \overline{b}) \in \mathcal{S}.$$

Theorem

Let $\mathbf{A}$ *be a countable* ω*-homogeneous structure. Then* $\mathrm{Aut}(\mathbf{A})$ *admits a maximal metric if and only if there is a finite tuple* $\overline{a}$ *in* $\mathbf{A}$ *satisfying the following two requirements.*

Theorem

Let $\mathbf{A}$ *be a countable* ω*-homogeneous structure. Then* $\mathrm{Aut}(\mathbf{A})$ *admits a maximal metric if and only if there is a finite tuple* $\overline{a}$ *in* $\mathbf{A}$ *satisfying the following two requirements.*

1. *There is a finite set* $\mathcal{R}$ *of parameter-free types so that* $\mathbf{X}_{\overline{a},\mathcal{R}}$ *is connected,*

Theorem

Let $\mathbf{A}$ *be a countable* ω*-homogeneous structure. Then* $\mathrm{Aut}(\mathbf{A})$ *admits a maximal metric if and only if there is a finite tuple* $\overline{a}$ *in* $\mathbf{A}$ *satisfying the following two requirements.*

1. *There is a finite set* $\mathcal{R}$ *of parameter-free types so that* $\mathbf{X}_{\overline{a},\mathcal{R}}$ *is connected, and*
2. *for every tuple* $\overline{b}$ *extending* $\overline{a}$*, there is a finite set* $\mathcal{S}$ *of parameter-free types so that*

$$\{\overline{c} \in \mathcal{O}(\overline{b}) \mid \overline{c} \text{ extends } \overline{a}\}$$

has finite diameter in the graph $\mathbf{X}_{\overline{b},\mathcal{S}}$*.*

Theorem

Let $\mathbf{A}$ *be a countable* ω*-homogeneous structure. Then* $\mathrm{Aut}(\mathbf{A})$ *admits a maximal metric if and only if there is a finite tuple* $\bar{a}$ *in* $\mathbf{A}$ *satisfying the following two requirements.*

1. *There is a finite set* $\mathcal{R}$ *of parameter-free types so that* $\mathbf{X}_{\bar{a},\mathcal{R}}$ *is connected, and*
2. *for every tuple* $\bar{b}$ *extending* $\bar{a}$*, there is a finite set* $\mathcal{S}$ *of parameter-free types so that*
$$\{\bar{c} \in \mathcal{O}(\bar{b}) \mid \bar{c} \text{ extends } \bar{a}\}$$
has finite diameter in the graph $\mathbf{X}_{\bar{b},\mathcal{S}}$*.*

Condition (2), which in itself is equivalent to $\mathrm{Aut}(\mathbf{A})$ being locally (OB), may require some amount of work to verify.

Theorem (Milnor–Švarc Theorem)

For $\overline{a}$ and $\mathcal{R}$ as above, the map

$$g \in \mathrm{Aut}(\mathbf{A}) \mapsto g \cdot \overline{a} \in \mathbf{X}_{\overline{a},\mathcal{R}}$$

is a quasi-isometry between $\mathrm{Aut}(\mathbf{A})$ *and* $\mathbf{X}_{\overline{a},\mathcal{R}}$.

Theorem (Milnor–Švarc Theorem)

For $\overline{a}$ and $\mathcal{R}$ as above, the map

$$g \in \mathrm{Aut}(\mathbf{A}) \mapsto g \cdot \overline{a} \in \mathbf{X}_{\overline{a},\mathcal{R}}$$

is a quasi-isometry between $\mathrm{Aut}(\mathbf{A})$ and $\mathbf{X}_{\overline{a},\mathcal{R}}$.

As an application of this, let $\mathbf{T}$ denote the $\aleph_0$-regular unrooted tree.

Theorem (Milnor–Švarc Theorem)

For $\bar{a}$ and $\mathcal{R}$ as above, the map

$$g \in \mathrm{Aut}(\mathbf{A}) \mapsto g \cdot \bar{a} \in \mathbf{X}_{\bar{a},\mathcal{R}}$$

is a quasi-isometry between $\mathrm{Aut}(\mathbf{A})$ *and* $\mathbf{X}_{\bar{a},\mathcal{R}}$.

As an application of this, let $\mathbf{T}$ denote the $\aleph_0$-regular unrooted tree.

Then, if a is a single vertex and $\mathcal{R} = \{E\}$ consist of the single type which is the edge relation, Conditions (1) and (2) are verified.

Theorem (Milnor–Švarc Theorem)

For $\overline{a}$ and $\mathcal{R}$ as above, the map

$$g \in \mathrm{Aut}(\mathbf{A}) \mapsto g \cdot \overline{a} \in \mathbf{X}_{\overline{a},\mathcal{R}}$$

is a quasi-isometry between $\mathrm{Aut}(\mathbf{A})$ and $\mathbf{X}_{\overline{a},\mathcal{R}}$.

As an application of this, let $\mathbf{T}$ denote the $\aleph_0$-regular unrooted tree.

Then, if a is a single vertex and $\mathcal{R} = \{E\}$ consist of the single type which is the edge relation, Conditions (1) and (2) are verified.

So

$$g \in \mathrm{Aut}(\mathbf{T}) \mapsto g(a) \in \mathbf{T}$$

is a quasi-isometry between $\mathrm{Aut}(\mathbf{T})$ and $\mathbf{X}_{a,\mathcal{R}} = \mathbf{T}$.

The verification that $\mathrm{Aut}(\mathbf{A})$ is locally (OB) often relies on identifying an appropriate independence relation $\perp_{\overline{a}}$ between finite subsets of $\mathbf{A}$ relative to a fixed finite tuple $\overline{a}$ in $\mathbf{A}$.

The verification that $\mathrm{Aut}(\mathbf{A})$ is locally (OB) often relies on identifying an appropriate independence relation $\mathop{\smile\!\!\!\!\!\!|}_{\bar{a}}$ between finite subsets of $\mathbf{A}$ relative to a fixed finite tuple $\bar{a}$ in $\mathbf{A}$.

The requirements on $\mathop{\smile\!\!\!\!\!\!|}_{\bar{a}}$ are the model theoretical conditions of *symmetry*, *monotonicity*, *existence* and *stationarity*.

The verification that $\mathrm{Aut}(\mathbf{A})$ is locally (OB) often relies on identifying an appropriate independence relation $\mathop{\smile\!\!\!\!\!\!|}_{\bar{a}}$ between finite subsets of $\mathbf{A}$ relative to a fixed finite tuple $\bar{a}$ in $\mathbf{A}$.

The requirements on $\mathop{\smile\!\!\!\!\!\!|}_{\bar{a}}$ are the model theoretical conditions of *symmetry*, *monotonicity*, *existence* and *stationarity*.

K. Tent and M. Ziegler have shown that similar notions appear in Fraïssé classes provided these admit a canonical amalgamation scheme.

The verification that $\mathrm{Aut}(\mathbf{A})$ is locally (OB) often relies on identifying an appropriate independence relation $\mathop{\smile\!\!\!\!\!\!\,|}_{\overline{a}}$ between finite subsets of $\mathbf{A}$ relative to a fixed finite tuple $\overline{a}$ in $\mathbf{A}$.

The requirements on $\mathop{\smile\!\!\!\!\!\!\,|}_{\overline{a}}$ are the model theoretical conditions of *symmetry*, *monotonicity*, *existence* and *stationarity*.

K. Tent and M. Ziegler have shown that similar notions appear in Fraïssé classes provided these admit a canonical amalgamation scheme.

For our purposes, we require somewhat more.

The verification that $\mathrm{Aut}(\mathbf{A})$ is locally (OB) often relies on identifying an appropriate independence relation $\downfree_{\bar{a}}$ between finite subsets of $\mathbf{A}$ relative to a fixed finite tuple $\bar{a}$ in $\mathbf{A}$.

The requirements on $\downfree_{\bar{a}}$ are the model theoretical conditions of *symmetry*, *monotonicity*, *existence* and *stationarity*.

K. Tent and M. Ziegler have shown that similar notions appear in Fraïssé classes provided these admit a canonical amalgamation scheme.

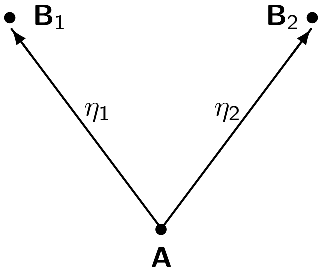
For our purposes, we require somewhat more.

Definition

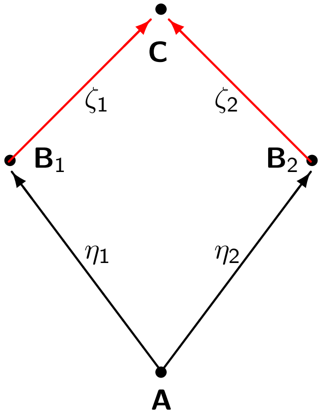
Given an Fraïssé class $\mathcal{K}$ with limit $\mathbf{K}$ and a finitely generated substructure $\mathbf{A} \subseteq \mathbf{K}$, we say that $\mathcal{K}$ satifies functorial amalgamation over $\mathbf{A}$ if there is a way of choosing the amalgamations over $\mathbf{A}$ in the class $\mathcal{K}$ to be functorial with respect to embeddings.

In terms of arrows

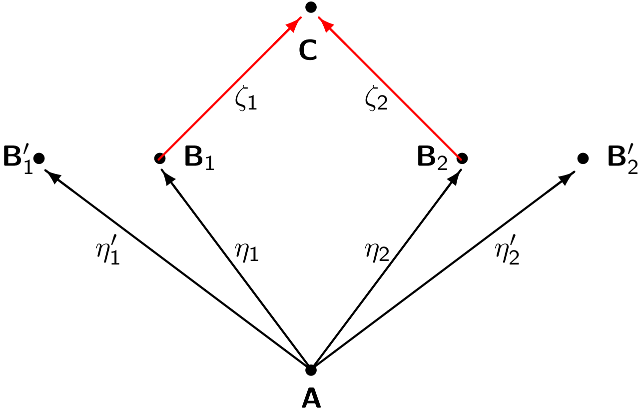
In terms of arrows



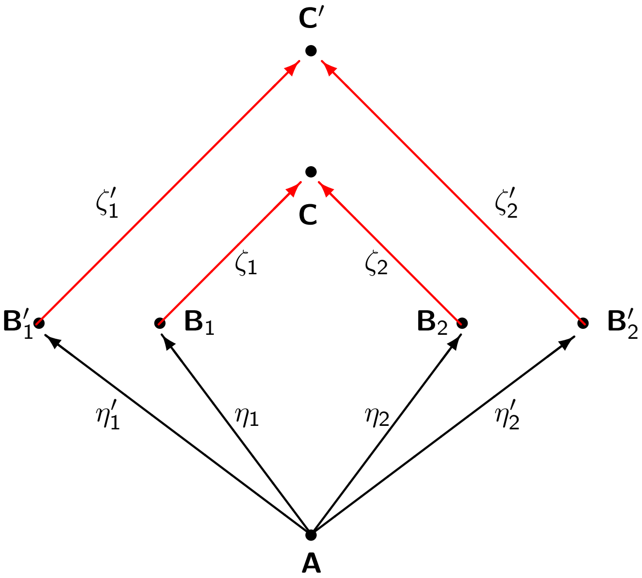
In terms of arrows



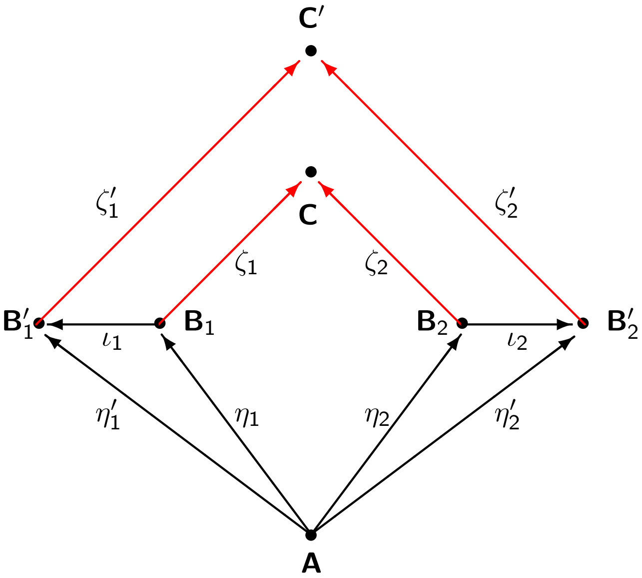
In terms of arrows



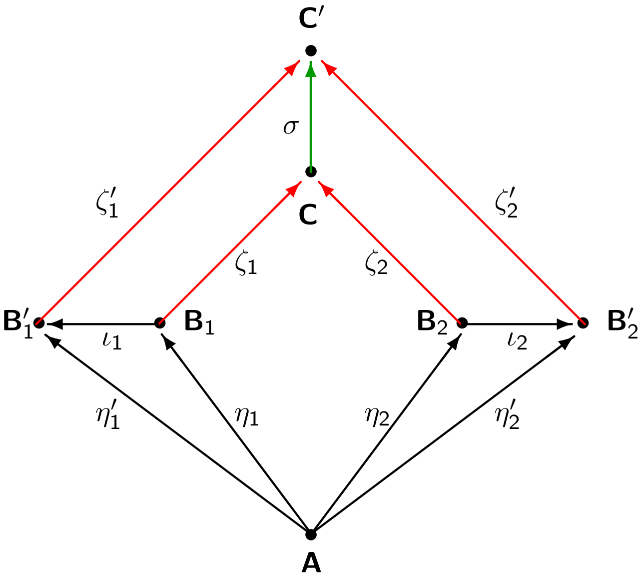
In terms of arrows



In terms of arrows



In terms of arrows



For example, the Fraïssé class of finite metric spaces admits a functorial amalgamation over a single point, namely the free amalgamation.

For example, the Fraïssé class of finite metric spaces admits a functorial amalgamation over a single point, namely the free amalgamation.

Theorem

Suppose $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$ and assume that $\mathbf{A}$ is a finitely generated substructure of $\mathbf{K}$ so that $\mathcal{K}$ admits a functorial amalgamation over $\mathbf{A}$.

For example, the Fraïssé class of finite metric spaces admits a functorial amalgamation over a single point, namely the free amalgamation.

Theorem

Suppose $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$ *and assume that* $\mathbf{A}$ *is a finitely generated substructure of* $\mathbf{K}$ *so that $\mathcal{K}$ admits a functorial amalgamation over* $\mathbf{A}$.
Then $\mathrm{Aut}(\mathbf{K})$ is locally (OB).

For example, the Fraïssé class of finite metric spaces admits a functorial amalgamation over a single point, namely the free amalgamation.

Theorem

Suppose $\mathcal{K}$ is a Fraïssé class with limit $\mathbf{K}$ and assume that $\mathbf{A}$ is a finitely generated substructure of $\mathbf{K}$ so that $\mathcal{K}$ admits a functorial amalgamation over $\mathbf{A}$.
Then $\mathrm{Aut}(\mathbf{K})$ is locally (OB).

Using this, we may show that, for any fixed point $p \in \mathbb{QU}$, the map

$$g \in \mathrm{Isom}(\mathbb{QU}) \mapsto g(p) \in \mathbb{QU}$$

is a quasi-isometry.

In many familiar cases, though we are able to identify the large scale geometry of a topological group, it turns out that this is trivial.

In many familiar cases, though we are able to identify the large scale geometry of a topological group, it turns out that this is trivial.

Theorem (P. Cameron)

Let $\mathbf{A}$ *be an* $\aleph_0$*-categorical countable structure.*
Then $\mathrm{Aut}(\mathbf{A})$ *is quasi-isometric to a point.*

In many familiar cases, though we are able to identify the large scale geometry of a topological group, it turns out that this is trivial.

Theorem (P. Cameron)

Let $\mathbf{A}$ *be an* $\aleph_0$*-categorical countable structure.*
Then $\mathrm{Aut}(\mathbf{A})$ *is quasi-isometric to a point.*

Similarly, using forking calculus, we may show the same conclusion for certain stable structures.

In many familiar cases, though we are able to identify the large scale geometry of a topological group, it turns out that this is trivial.

Theorem (P. Cameron)

Let **A** *be an* $\aleph_0$*-categorical countable structure.*
Then $\mathrm{Aut}(\mathbf{A})$ *is quasi-isometric to a point.*

Similarly, using forking calculus, we may show the same conclusion for certain stable structures.

Theorem

Let **A** *be a saturated countable model of an* ω*-stable theory.*
Then $\mathrm{Aut}(\mathbf{A})$ *is quasi-isometric to a point.*

The stability of the underlying structure is similarly reflected in the large scale geometry in the absence of saturation.

The stability of the underlying structure is similarly reflected in the large scale geometry in the absence of saturation.

Theorem

Suppose $\mathbf{M}$ *is a countable atomic model of a stable theory* T *so that* $\mathrm{Aut}(\mathbf{M})$ *is locally (OB). Then* $\mathrm{Aut}(\mathbf{M})$ *admits a compatible left-invariant coarsely proper stable metric.*

The stability of the underlying structure is similarly reflected in the large scale geometry in the absence of saturation.

Theorem

Suppose $\mathbf{M}$ *is a countable atomic model of a stable theory* T *so that* $\mathrm{Aut}(\mathbf{M})$ *is locally (OB). Then* $\mathrm{Aut}(\mathbf{M})$ *admits a compatible left-invariant coarsely proper stable metric.*
It follows that $\mathrm{Aut}(\mathbf{M})$ *has a coarsely proper continuous affine isometric action on a reflexive Banach space.*

The stability of the underlying structure is similarly reflected in the large scale geometry in the absence of saturation.

Theorem

Suppose $\mathbf{M}$ *is a countable atomic model of a stable theory* T *so that* $\mathrm{Aut}(\mathbf{M})$ *is locally (OB). Then* $\mathrm{Aut}(\mathbf{M})$ *admits a compatible left-invariant coarsely proper stable metric.*
It follows that $\mathrm{Aut}(\mathbf{M})$ *has a coarsely proper continuous affine isometric action on a reflexive Banach space.*

Motivated by the preceding results, one could be hopeful that the assumption that $\mathrm{Aut}(\mathbf{M})$ be locally (OB) would be superflous.

The stability of the underlying structure is similarly reflected in the large scale geometry in the absence of saturation.

Theorem

Suppose $\mathbf{M}$ *is a countable atomic model of a stable theory* T *so that* $\mathrm{Aut}(\mathbf{M})$ *is locally (OB). Then* $\mathrm{Aut}(\mathbf{M})$ *admits a compatible left-invariant coarsely proper stable metric.*
It follows that $\mathrm{Aut}(\mathbf{M})$ *has a coarsely proper continuous affine isometric action on a reflexive Banach space.*

Motivated by the preceding results, one could be hopeful that the assumption that $\mathrm{Aut}(\mathbf{M})$ be locally (OB) would be superflous.

However, this is not so.

Theorem (J. Zielinski)

There is an atomic model $\mathbf{M}$ *of an* ω*-stable theory so that* $\mathrm{Aut}(\mathbf{M})$ *is not locally (OB).*